4. RVACHEV V.L. and PROTSENKO V.S., Contact Problems of Elasticity Theory for Non-Classical Domains. Naukova Dumka, Kiev, 1977.
5. POPOV G. YA., Elastic Stress Concentration around Stamps, Slits, Thin Inclusions and Reinforcements, Nauka, Moscow, 1982.
6. KRASNOSEL'SKII M.A., ZABREIKO P.P., PUSTYL'NIK E.I. and SOBOLEVSKII P.E., Integral Operators in Spaces of Summable Functions. Nauka, Moscow, 1966.
7. COURANT R. and HILBERT D., Methods of Mathematical Physics /Russian translation/, Vol.1, Gostekhteorizdat, Moscow-Leningrad, 1951.
8. BORODACHEV A.N., Determination of the stress intensity factor for a plane elliptical crack for arbitrary boundary conditions, Izv. Akad. Nauk SSSR, Mekhan. Tverd Tela, No.2, 1981.
9. SIH G. and LIEBOWITZ G., Mathematical theory of brittle fracture. Fracture /Russian translation/, Vol. 2, Mir, Moscow, 1975.
10. ROSTOVTSEV N.A., On the elasticity theory of an inhomogeneous medium, PMM, Vol.28, No. 4, 1984.
11. ARUTIUNYAN N. KH., Plane contact problem of creep theory, PMM, vol. 23, no. 5, 1959.
12. ALEKSANDROV V.M. and SUMBATIAN M.A., On a solution of the contact problem of non-linear steady creep for a half-plane, Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, No.1, 1983.

Translated by M.D.F.

PMM U.S.S.R., Vol.48, No.5, pp. 611-618, 1984.
Printed in Great Britain

0021-8928/84 $10.00+0.00

A SPECIAL RELATIONSHIP IN SPHEROIDAL WAVE FUNCTIONS AND ITS APPLICATION TO CONTACT PROBLEMS*

S.M. MKHITARYAN

A spectral and kindred relationship are set up by methods of the theory of the generalized potential /1/ for an integral operator generated by a symmetric difference kernel in the form of a Macdonald function in two identical semi-infinite intervals $\{(-\infty, -a), (a, \infty)\}$ that contain spheroidal wave functions. The formula for the expansion of an arbitrary function in these functions is also set up by a well-known method /2/. On the basis of the results obtained, a solution is then constructed for the integral equation of the contact problem of the impression of two identical stamps with half-plane bases into a half-space being deformed in a power-law form in the formulation of /3/.

This contact problem can be described by the same integral equation when the elastic modulus of a linearly elastic half-space changes with depth according to a power law /1/.

The spectral relationships in classical orthogonal polynomials for extensive classes of integral operators in mathematical physics are presented in /4,5/, where the method of orthogonal polynomials based on them is also elucidated, and numerous applications of this method are given to contact and mixed problems of elasticity theory. We also mention /6-9/ which are related directly to the investigation presented here.

1. Consider the integral equation

$$K\varphi_s = f_s(y), \quad K\varphi_s = \left(\int_{-\infty}^{-a} + \int_a^{\infty}\right) \frac{K_\mu(|s||y-\eta|)}{|y-\eta|^\mu}\varphi_s(\eta)\,d\eta, \quad |\mu| < \frac{1}{2} \tag{1.1}$$

in order to set up a spectral relationship for the integral operator $\psi_s(y) = K\varphi_s$, where $K_\mu(y)$ is the Macdonald function. To this end, following /8/, we introduce the function ($\Gamma(x)$ is the gamma function)

$$V(y, z, s) = U_s(y, z) = \int_{-\infty}^{\infty} U(x, y, z)\, e^{isx}\, dx = \tag{1.2}$$

*Prikl. Matem. Mekhan., Vol. 48, 5, 845-853, 1984

$$\frac{\sqrt{\pi}\,|s|^{\mu}}{2^{\mu-1}\Gamma(\mu+{}^1/_2)}\left(\int\limits_{-\infty}^{-a}+\int\limits_{a}^{\infty}\right)\frac{K_\mu(|s|\sqrt{(y-\eta)^2-z^2})}{[(y-\eta)^2+z^2]^{\mu/2}}\varphi_s(\eta)\,d\eta$$

which is the Fourier transform in the variable x for the generalized potential

$$U(x,y,z)=\iint\limits_{\omega}\frac{\varphi(\xi,\eta)\,d\xi\,d\eta}{[(x-\xi)^2+(y-\eta)^2+z^2]^{\mu+1/2}}$$

$$\omega=\{z=0;\ |x|<\infty,\ |y|>a\}$$

which has a finite source power.

It follows from /1,8/ that integral equation (1.1) is equivalent to the following boundary value problem:

$$\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}+\frac{2\mu}{z}\frac{\partial V}{\partial z}-s^2V=0,\quad (y,z)\,\overline{\in}\,L \tag{1.3}$$

$$V(y,z,s)|_{z=0}=g_s(y),\ (y,0)\in L;\ V(y,z,s)\to 0,\ y^2+z^2\to\infty$$

$$g_s(y)=\sqrt{\pi}2^{1-\mu}|s|^{\mu}[\Gamma(\mu+{}^1/_2)]^{-1}f_s(y)$$

for the whole $y0z$ plane with the disconnected rays $L=\{z=0;\ -\infty<y\leqslant -a,\ a\leqslant y<\infty\}$.

After the solution of the boundary value problem (1.3) has been constructed, the source density, i.e., the solution of (1.1), is determined from the formula /8/

$$-2\pi\varphi_s(y)=\lim_{z\to\infty}\left[\operatorname{sgn} z\,|z|^{2\mu}\frac{\partial V}{\partial z}\right],\quad (y,0)\in L \tag{1.4}$$

We construct the solution of the boundary value problem (1.3) by separation of variables. To use the available results from /10/ for this purpose, we set

$$V(y,z,s)=|z|^{-\mu}W(y,z,s) \tag{1.5}$$

Then the differential equation from (1.3) is converted into the following:

$$\frac{\partial^2 W}{\partial y^2}+\frac{\partial^2 W}{\partial z^2}+\mu(1-\mu)\frac{W}{z^2}-s^2W=0 \tag{1.6}$$

Furthermore, we introduce the elliptic coordinates (/10/, p. 136)

$$w=y+iz=a\operatorname{ch}\zeta,\quad \zeta=u+iv,\ |u|<\infty,\ 0\leqslant v\leqslant\pi \tag{1.7}$$

$$y=a\operatorname{ch}u\cos v,\ z=a\operatorname{sh}u\sin v$$

By using the conformal mapping (1.7), the complex w plane with the slit L is evidently mapped into the strip $\Pi=\{-\infty<u<\infty,0\leqslant v\leqslant\pi\}$ where the line $v=0$ corresponds to the twice-covered ray $y\geqslant a$, while the line $v=\pi$ corresponds to the twice-covered ray $y\leqslant -a$ of the w plane. Taking (1.7) into account we now set

$$W(y,z,s)=W(a\operatorname{ch}u\cos v,\ a\operatorname{sh}u\sin v,s)=W_0(u,v)=F(u)G(v) \tag{1.8}$$

Using the results from /10/, after certain elementary manipulations we reduce the partial differential equation (1.6) to the following two ordinary differential equations

$$F''+[\beta-2q\operatorname{ch}2u+\mu(1-\mu)\operatorname{sh}^{-2}u]F=0,\ -\infty<u<\infty \tag{1.9}$$

$$G''-[\beta-2q\cos 2v-\mu(1-\mu)\sin^{-2}v]G=0,\ 0<v<\pi, \tag{1.10}$$

$$q=a^2s^2/4$$

where β is the separation parameter. We note that if we set $v=iu$ formally, then (1.10) reduces to (1.9). Hence, we can limit ourselves to one of them, for instance, (1.10).

By using the substitution $G(v)=\sqrt{\sin v}\,H(v)$ we convert (1.10) into a differential equation for spheroidal wave functions (/10/, p. 170)

$$\frac{d^2H}{dv^2}+\operatorname{ctg}v\frac{dH}{dv}+[\lambda+4\theta\sin^2 v-\varkappa^2\sin^{-2}v]H=0 \tag{1.11}$$

$$\lambda=-\beta-{}^1/_4-2\theta=-\beta(\theta)-{}^1/_4-2\theta,\ \theta=-q,\ \varkappa={}^1/_2-\mu$$

Equation (1.11) and its solution are examined in detail in /11,12/.

The functions

$$P^{\varkappa}_{\nu}(\cos v,\theta)=\sum_{r=-\infty}^{\infty}(-1)^r a^{\varkappa}_{\nu,r}(\theta)P^{\varkappa}_{\nu+2r}(\cos v),\quad 0<v<\pi \tag{1.12}$$

$$Q_{s_\nu}^{\varkappa}(\cos v, \theta) = \sum_{r=-\infty}^{\infty} (-1)^r a_{\nu, r}^{\varkappa}(\theta) Q_{\nu+2r}^{\varkappa}(\cos v)$$

are two linearly independent solutions of (1.11), where ν is the characteristic index of (1.11), and $\lambda = \lambda_\nu^\varkappa(\theta)$ and $\lambda_\nu^\varkappa(0) = \nu(\nu+1)$, $P_\nu^\varkappa(x)$ and $Q_\nu^\varkappa(x)$ are Legendre functions of the first and second kinds, respectively, while the coefficients $a_{\nu,r}^\varkappa(\theta)$ are determined from trinomial recursion relationships (/10/, p. 171). These coefficients are determined such that

$$a_{\nu,0}^\varkappa(\theta) = a_{-\nu-1,0}^\varkappa(\theta) = a_{\nu,0}^{-\varkappa}(\theta), \quad a_{\nu,0}^\varkappa(0) = 1$$

To determine the characteristic index ν, we note the following. This parameter is determined by the value $\theta = 0$ since $\lambda_\nu^\varkappa(0) = \nu(\nu+1)$. Consequently, we set $\theta = 0$ in (1.11), whereupon it goes over into the Legendre differential equation examined in /9/, in which the plane analogue of the problem formulated here is discussed.

Therefore

$$\nu(\nu+1) = -\alpha - 1/4, \quad \alpha = \beta(0)$$

from which we have, taking appropriate results from /9/ into account,

$$\nu = -1/2 + i\sqrt{\alpha}, \quad \alpha > 0$$

For such values of ν we have from the above-mentioned trinomial recurrent relationships

$$\overline{a_{\nu,r}^\varkappa(\theta)} = a_{-\nu-1,r}^\varkappa(\theta) = a_{\nu,-r}^\varkappa(\theta) = a_{\bar\nu,r}^\varkappa(\theta) \tag{1.13}$$

For a complete determination of these coefficients, we normalize them by the conditions (/12/, p. 286)

$$\sum_{r=-\infty}^{\infty} \frac{\Gamma(\nu+\varkappa+2r+1)\,\Gamma(\nu-\varkappa+1)\,[a_{\nu,r}^\varkappa(\theta)]^2}{\Gamma(\nu-\varkappa+2r+1)\,\Gamma(\nu+\varkappa+1)\,[2(\nu+2r)+1]} = \frac{1}{2\nu+1}, \quad a_{\nu,0}^\varkappa(\theta) > 0$$

Starting from the above, we represent the solution of (1.10) in the form

$$G(v) = \sqrt{\sin v}\,[APs_\nu^\varkappa(\cos v, \theta) + BQs_\nu^\varkappa(\cos v, 0)], \; 0 < v < \pi$$

and construct its solution: even $(G_{\nu,\varkappa}^+(\pi - v) = G_{\nu,\varkappa}^+(v),\ 0 < v < \pi)$ and odd $(G_{\nu,\varkappa}^-(\pi - v) = -G_{\nu,\varkappa}^-(v),\ 0 < v < \pi)$ relative to the point $v = \pi/2$. Using the formulas (/12/, p. 287) as in /9/, we will have

$$G_{\nu,\varkappa}^{\pm}(v) = \sqrt{\sin v}\left\{P_{s_\nu}^\varkappa(\cos v, \theta) \pm \frac{2}{\pi}\,\mathrm{Re}\left[\begin{matrix}\mathrm{tg}(\pi\delta)\\ \mathrm{ctg}(\pi\delta)\end{matrix}\, Q_{s_\nu}^\varkappa(\cos v, \theta)\right]\right\}, \quad 0 < v < \pi \tag{1.14}$$

$$\nu = -1/2 + i\sqrt{\alpha}, \quad \alpha = \tau^2, \quad \tau > 0, \quad \varkappa = 1/2 - \mu, \quad \delta = (\mu - i\tau)/2$$

where according to (1.13) it follows from (1.12) that

$$\overline{P_{s_\nu}^\varkappa(\cos v, \theta)} = P_{s_\nu}^\varkappa(\cos v, \theta), \quad \overline{Q_{s_\nu}^\varkappa(\cos v, \theta)} = Q_{s_{\bar\nu}}^\varkappa(\cos v, \theta)$$

We now turn to (1.9). The unique solution of this equation that is bounded on the axis $-\infty < u < \infty$ and vanishes at infinity, has the form /10-12/

$$F_\nu^\varkappa(u) = \sqrt{|\mathrm{sh}\,u|}\, S_\nu^{\varkappa(3)}(\mathrm{ch}\,u, \theta), \quad -\infty < u < \infty \tag{1.15}$$

where $S_\nu^{\varkappa(3)}(z, \theta)$ is a spheroidal wave function of the third kind. The following representation can be obtained for this function

$$S_\nu^{\varkappa(3)}(\mathrm{ch}\,u, \theta) = l_\nu^\varkappa(\theta)\,|\mathrm{th}\,u|^{-\varkappa}(\mathrm{ch}\,u)^{-1/2} \times \tag{1.16}$$

$$\sum_{r=-\infty}^{\infty} (-1)^r a_{\nu,r}^\varkappa(\theta)\, K_{i\tau+2r}(2\sqrt{q}\,\mathrm{ch}\,u) \quad (-\infty < u < \infty)$$

$$l_\nu^\varkappa(\theta) = (\pi^2 q)^{-1/4} \exp[\pi(2\tau - 3i)/4]\, s_\nu^\varkappa(\theta), \quad q = a^2 s^2/4$$

$$s_\nu^\varkappa(\theta) = \left[\sum_{r=-\infty}^{\infty} (-1)^r a_{\nu,r}^\varkappa(\theta)\right]^{-1}, \quad \nu = -\frac{1}{2} + i\tau, \quad \tau > 0$$

Using (1.13) it can be shown that

$$\overline{S_\nu^{\varkappa(3)}(\mathrm{ch}\,u, \theta)} = -iS_\nu^{\varkappa(3)}(\mathrm{ch}\,u, \theta) \tag{1.17}$$

Therefore, according to (1.5) and (1.8), the boundary value problem (1.3) has a normal solution

$$V(y, z, s) = (|\operatorname{sh} u| \sin v)^{-\mu} F_\nu^\varkappa(u)\, G_{\nu,\varkappa}^{\pm}(v) \qquad (1.18)$$
$$(-\infty < u < \infty,\ 0 \leqslant v \leqslant \pi)$$

which is bounded in the strip $\Pi = \{-\infty < u < \infty,\ 0 \leqslant v \leqslant \pi\}$ and vanishes at infinity, where the functions $F_\nu^\varkappa(u)$ are expressed by (1.15), and (1.16), the functions $G_{\nu,\varkappa}^{\pm}(v)$ by (1.14), and the variables $y, z; u, v$ are connected by the dependence (1.7).

To calculate the source density corresponding to the potential (1.18), we use the representations of the functions $P_\nu^\varkappa(\cos v)$ and $Q_\nu^\varkappa(\cos v)$ in terms of the hypergeometric functions (/13/, p. 144-148) and by virtue of the dependences (1.7), we convert (1.4) to the form

$$-2\pi\varphi_s(y) = (a \operatorname{sh} u)^{2\mu-1} \lim_{v\to 0} (\sin v)^{2\mu} \frac{\partial V}{\partial v}, \quad 0 < u < \infty$$

Furthermore, proceeding exactly as in /9/ and taking (1.2) into account, after some reduction we arrive at the spectral relationship

$$\int_a^\infty \left\{\frac{K_\mu(|s||y-\eta|)}{|y-\eta|^\mu} \pm \frac{K_\mu[|s|(y+\eta)]}{(y+\eta)^\mu}\right\} (\eta^2/a^2-1)^{-\varkappa/2} S_\nu^{\varkappa(3)}(\eta/a, \theta)\, d\eta \equiv \qquad (1.19)$$
$$I_\mu^\pm(y, s) = \lambda_{\nu,\varkappa}^\pm (y^2/a^2-1)^{\varkappa/2} S_\nu^{\varkappa/3}(y/a, \theta), \quad y > a$$
$$\lambda_{\nu,\varkappa}^\pm = \pi^{3/2} a^{2\varkappa} (2|s|)^{-\mu} k_{\nu,\varkappa}^\pm(\theta) [l_{\nu,\varkappa}^\pm(\theta)]^{-1}, \quad \nu = -1/2 + i\tau, \quad \tau > 0$$
$$k_{\nu,\varkappa}^\pm(\theta) = [\Gamma(1-\varkappa)]^{-1} \sum_{r=-\infty}^{\infty} (-1)^r \operatorname{Re}[a_{\nu,r}^\varkappa(\theta)] \pm$$
$$\pi^{-1} \sin(\pi\mu)\,\Gamma(\varkappa) \operatorname{Re}\left[\begin{matrix}\operatorname{tg}(\pi\delta)\\ \operatorname{ctg}(\pi\delta)\end{matrix} \sum_{r=-\infty}^{\infty} (-1)^r a_{\nu,r}^\varkappa(\theta)\right], \quad \delta = \frac{\mu - i\tau}{2}$$
$$l_{\nu,\varkappa}^\pm(\theta) = \pm \operatorname{Re}\left[\begin{matrix}\operatorname{tg}(\pi\delta)\\ \operatorname{ctg}(\pi\delta)\end{matrix} \sum_{r=-\infty}^{\infty} (-1)^r a_{\nu,r}^\varkappa(\theta) \frac{\Gamma(1-2\delta+2r)}{\Gamma(2\bar\delta+2r)}\right]$$

We also obtain the integral relations related to (1.19) when $0 < y < a$. The line $u = 0$ $(0 < v < \pi/2)$ corresponds to this interval. It is therefore necessary to calculate the value $[(\operatorname{sh} u)^{-\varkappa} F_\nu^\varkappa(u)]_{+0}$. To this end we use the well-known relationships between the spheroidal wave functions (/10/, pp. 173-175), which yield

$$S_\nu^{\varkappa(3)}(\operatorname{ch} u, \theta) = [\pi \operatorname{sh}(\pi\tau)]^{-1} \exp[\pi(2\tau + i\mu)] \times \qquad (1.20)$$
$$\{\sin(2\pi\bar\delta)\, K_\nu^\varkappa(\theta)\, Qs_{-\nu-1}^\varkappa(\operatorname{ch} u, \theta) -$$
$$i \sin(2\pi\delta)\, \overline{K_\nu^\varkappa(\theta)}\, Qs_\nu^\varkappa(\operatorname{ch} u, \theta)\}, \quad -\infty < u < \infty$$
$$K_\nu^\varkappa(\theta) = 2^{-1}(q/4)^{\nu/2}\, \Gamma(2\bar\delta) \exp[-3\pi(2\tau + i)/4] \times s_\nu^\varkappa(\theta)\, L_\nu^\varkappa(\theta)\, \overline{[L_\nu^\varkappa(\theta)]}^{-1}$$
$$L_\nu^\varkappa(\theta) = \sum_{r=0}^{\infty} \frac{(-1)^r a_{\nu,r}^\varkappa(\theta)}{r!\,\Gamma(1+i\tau-r)}$$
$$(\nu = -1/2 + i\tau,\ \ \tau > 0,\ \ \delta = (\mu - i\tau)/2,\ -\nu - 1 = \bar\nu,\ \ \varkappa = 1/2 - \mu)$$

Furthermore, taking account of (1.14)-(1.16) and (1.20), we obtain the following integral relationships again by using (1.2) and (1.18)

$$I_\mu^\pm(y, s) = h_{\nu,\varkappa}^\pm (1 - y^2/a^2)^{\varkappa/2} H_{\nu,\varkappa}^\pm[\arccos(y/a)], \ 0 < y < a \qquad (1.21)$$
$$h_{\nu,\varkappa}^\pm = \pm 2^{-1/2}\pi^{3/2} a^{2\varkappa} |s|^{-\mu} M_\tau \left\{\operatorname{Re}\left[\begin{matrix}\operatorname{tg}(\pi\delta)\\ \operatorname{ctg}(\pi\delta)\end{matrix} \sum_{r=0}^{\infty} (-1)^r a_{\nu,r}^\varkappa(\theta) \times\right.\right.$$
$$\left.\left.\Gamma(1-2\delta+2r)(\Gamma(2\bar\delta+2r))^{-1}\right]\right\}^{-1}, \quad H_{\nu,\varkappa}^\pm(v) = (\sin v)^{-1/2} G_{\nu,\varkappa}^\pm(v)$$
$$M_\tau = 2^{\varkappa-1}[\pi \operatorname{sh}(\pi\tau)]^{-1} \exp(2\pi\tau)\, \Gamma(\varkappa)\, s_\nu^\varkappa(\theta) \times$$
$$[\sin(2\pi\delta)\, \overline{K_\nu^\varkappa(\theta)} + i \sin(2\pi\bar\delta)\, K_\nu^\varkappa(\theta)]$$

Evidently $\bar M_\tau = -iM_\tau$. In place of (1.17) this ensures the reality of relationships (1.19) and (1.21). Since $K_\mu(x)$ and $P_\nu^\mu(x)$ are analytic functions of the parameter μ, by the Schwartz symmetry principle, these relationships can be continued analytically in the band $|\operatorname{Re}\mu| < 1/2$.

Note that the relationships (1.21) can be utilized in contact and maxed problems of elasticity theory for calculating displacements of the foundation outside the stamps or the fracture stresses outside the slits in bulky bodies.

2. We now turn to the equation of expanding an arbitrary function in functions $S_\nu^{\mu(3)}$ $(\mathrm{ch}\, u,\ \theta)$ for which we consider the differential equation (1.9) in the interval $0 < u < \infty$ and we will treat the parameter $\alpha = \beta(0)$ as a complex parameter in the upper half-plane $\mathrm{Im}\, \alpha \geqslant 0$. Both ends of the interval under consideration are singular for this equation; hence, we take $u = b \neq 0$ as reference point. We take the functions $\sqrt{\mathrm{sh}\, u}\, Ps_\nu^{\varkappa}(\mathrm{ch}\, u,\ \theta)$ and $\sqrt{\mathrm{sh}\, u}\, Ps_\nu^{-\varkappa}(\mathrm{ch}\, u,\ \theta)$ as two linearly-independent solutions of this equation, and following /2/, we construct its solutions $\varphi(u, \alpha)$ and $\chi(u, \alpha)$ such that

$$\varphi(b, \alpha) = 0, \quad \varphi'(b, \alpha) = -1$$
$$\chi(b, \alpha) = 1, \quad \chi'(b, \alpha) = q_0$$

It is seen that

$$\varphi(u,\ \alpha) = \pi\,[2\sin(\pi\varkappa)]^{-1} s_\nu^{\varkappa}(\theta)\, s_\nu^{-\varkappa}(\theta) \sqrt{\mathrm{sh}\, b\, \mathrm{sh}\, u}\ \times$$
$$[Ps_\nu^{\varkappa}(\mathrm{ch}\, u,\ \theta)\, Ps_\nu^{-\varkappa}(\mathrm{ch}\, b,\ \theta) - Ps_\nu^{-\varkappa}(\mathrm{ch}\, u,\ \theta)\, Ps_\nu^{\varkappa}(\mathrm{ch}\, b,\ \theta)] \tag{2.1}$$
$$\chi(u,\ \alpha) = \pi\,[2\sin(\pi\varkappa)]^{-1} s_\nu^{\varkappa}(\theta)\, s_\nu^{-\varkappa}(\theta)\, \mathrm{sh}\, b \sqrt{\mathrm{sh}\, b\, \mathrm{sh}\, u}\ \times$$
$$[Ps_\nu^{\varkappa}(\mathrm{ch}\, u,\ \theta)\, Ps_\nu'^{-\varkappa}(\mathrm{ch}\, b,\ \theta) - Ps_\nu^{-\varkappa}(\mathrm{ch}\, u,\ \theta)\, Ps_\nu'^{\varkappa}(\mathrm{ch}\, b,\ \theta)]$$
$$(q_0 = {}^1/_2\, \mathrm{cth}\, b,\ 0 < u < \infty)$$

Furthermore, by using the well-known asymptotic representations (/10/, p. 177), we can write

$$Ps_\nu^{\varkappa}(\mathrm{ch}\, u,\ \theta) \sim [\Gamma(1-\varkappa)\, s_\nu^{\varkappa}(\theta)]^{-1}\ [\mathrm{sh}\,(u/2)]^{-\varkappa} + O\,\{[\mathrm{sh}\,(u/2)]^{2-\varkappa}\} \tag{2.2}$$
$$dPs_\nu^{\varkappa}(\mathrm{ch}\, u,\ \theta)/d\, \mathrm{ch}\, u \sim -\varkappa\, [4\Gamma(1-\varkappa)\, s_\nu^{\varkappa}(\theta)]^{-1}\ \times$$
$$[\mathrm{sh}\,(u/2)]^{-2-\varkappa} + O\,\{[\mathrm{sh}\,(u/2)^{-\varkappa}\},\ u \to 0$$

It therefore follows at once that all the solutions (1.9) belong to the space $L^2(0, b)$ for $\mathrm{Im}\sqrt{\alpha} > 0$, i.e., the case of the Weyl limit circle will hold at the point $u = 0$. The limit circle is the limit of the circles

$$l = -\frac{\chi(b_0, \alpha)\,\mathrm{ctg}\,\alpha_0 + \chi'(b_0, \alpha)}{\varphi(b_0, \alpha)\,\mathrm{ctg}\,\alpha_0 + \varphi'(b_0, \alpha)}$$

as $b_0 \to 0$.

Proceeding in exactly the same way as in /2/ (p. 95), and utilizing the asymptotic formulas (2.2), we find the limit circle $(|c| < \infty)$

$$m_1(\alpha) = \mathrm{sh}\, b\, \frac{cs_\nu^{-\varkappa}(\theta)\, Ps_\nu'^{-\varkappa}(\mathrm{ch}\, b,\ \theta) - s_\nu^{\varkappa}(\theta)\, Ps_\nu'^{\varkappa}(\mathrm{ch}\, b,\ \theta)}{s_\nu^{\varkappa}(\theta)\, Ps_\nu^{\varkappa}(\mathrm{ch}\, b,\ \theta) - cs_\nu^{-\varkappa}(\theta)\, Ps_\nu^{-\varkappa}(\mathrm{ch}\, b,\ \theta)} \tag{2.3}$$

Now, forming the function

$$\psi_1(u,\ \alpha) = \chi(u,\ \alpha) + m_1(\alpha)\, \varphi(u,\ \alpha)$$

from (2.1) and (2.3), we have after some reduction

$$\psi_1(u, \alpha) = \frac{\sqrt{\mathrm{sh}\, u}}{\sqrt{\mathrm{sh}\, b}} \cdot \frac{s_\nu^{\varkappa}(\theta)\, Ps_\nu^{\varkappa}(\mathrm{ch}\, u,\ \theta) - cs_\nu^{-\varkappa}(\theta)\, Ps_\nu^{-\varkappa}(\mathrm{ch}\, u,\ \theta)}{s_\nu^{\varkappa}(\theta)\, Ps_\nu^{\varkappa}(\mathrm{ch}\, b,\ \theta) - cs_\nu^{-\varkappa}(\theta)\, Ps_\nu^{-\varkappa}(\mathrm{ch}\, b,\ \theta)} \tag{2.4}$$

Turning to obtaining the needed solution of (1.9) in the interval (b, ∞), we note the following. In this case it is necessary to obtain a formula for expanding an arbitrary function in the functions $S_\nu^{\varkappa(3)}(\mathrm{ch}\, u, \theta)$ by forming the family of functions in the parameter ν or even the parameter α. But the spectral parameter α, which according to the method from /2/ is continued in the upper complex plane, is the value of the parameter β for $\theta = 0$, i.e., $\alpha = \beta(0)$. Consequently, unlike /2/, in the case selected it is necessary to take that solution of (1.9) which belongs to the space $L^2(b, \infty)$ for $\theta = 0$ and $\mathrm{Im}\sqrt{\alpha} > 0$. Since (1.9) goes over into the Legendre equation for $\theta = 0$, and its unique solution belonging to the space $L^2(b, \infty)$ for $\mathrm{Im}\sqrt{\alpha} > 0$ is the function $\sqrt{\mathrm{sh}\, u}\, Q_{-\nu-1}^{\varkappa}(\mathrm{ch}\, u)$, as results at once from the known asymptotic formula (/13/, p. 165), the needed solution of (1.9) will be the function $\sqrt{\mathrm{sh}\, u}\, Qs_{-\nu-1}^{\varkappa}(\mathrm{ch}\, u,\ \theta)$. It can be represented in the form /11/

$$\sqrt{\mathrm{sh}\, u}\, Qs_{-\nu-1}^{\varkappa}(\mathrm{ch}\, u,\ \theta) = \pi\,[2\sin(\pi\varkappa)]^{-1} \exp(i\pi\varkappa) \sqrt{\mathrm{sh}\, u}\ \times \tag{2.5}$$
$$\{Ps_\nu^{\varkappa}(\mathrm{ch}\, u,\ \theta) - \Gamma(\varkappa-\nu)\,[\Gamma(-\varkappa-\nu)]^{-1} Ps_\nu^{-\varkappa}(\mathrm{ch}\, u,\ \theta)\}$$
$$0 < u < \infty$$

Therefore, the function

$$\psi_2(u, \alpha) = \chi(u, \alpha) + m_2(\alpha)\varphi(u, \alpha)$$

can only differ from (2.5) by a constant factor. Taking account of (2.1) and (2.5) we hence find

$$m_2(\alpha) = \operatorname{sh} b \frac{Ps_\nu'^{\varkappa}(\operatorname{ch} b, \theta)\Gamma(-\varkappa-\nu) - Ps_\nu'^{-\varkappa}(\operatorname{ch} b, \theta)\Gamma(\varkappa-\nu)}{Ps_\nu^{-\varkappa}(\operatorname{ch} b, \theta)\Gamma(\varkappa-\nu) - Ps_\nu^{\varkappa}(\operatorname{ch} b, \theta)\Gamma(-\varkappa-\nu)} \tag{2.6}$$

Furthermore, by using (2.3) and (2.6), we calculate the spectral density /2/

$$\sigma_0(\tau) = \lim_{\operatorname{Im}\alpha\to+0}\{[m_2(\alpha) - m_1(\alpha)]^{-1}\} = \frac{2}{\pi}\rho(\tau)\operatorname{sh} b[\chi_\nu^{\varkappa}(b, \theta)]^2$$

$$\chi_\nu^{\varkappa}(u, \theta) = s_\nu^{\varkappa}(\theta) Ps_\nu^{\varkappa}(\operatorname{ch} u, \theta) - cs_\nu^{-\varkappa}(\theta) Ps_\nu^{-\varkappa}(\operatorname{ch} u, \theta)$$

Omitting the intermediate computations, we present the final result $(\alpha = \tau^2)$

$$\begin{aligned}\rho(\tau) = \pi \operatorname{sh}(\pi\tau)\{[c^2\gamma_\nu^{\varkappa}(\theta) \mid \Gamma(1/2 - \varkappa + i\tau)\mid^2 + \\ [\gamma_\nu^{\varkappa}(\theta)]^{-1} \mid \Gamma(1/2 + \varkappa + i\tau)\mid^2] \mid \cos[\pi(\varkappa + i\tau)]\mid^2 - \\ 2\pi c\cos(\pi\varkappa)\operatorname{ch}\pi\tau\}^{-1}, \quad \tau > 0, \; \varkappa = 1/2 - \mu\end{aligned} \tag{2.7}$$

$$\rho(\tau) = 0, \; \tau = it \; (t > 0), \quad \gamma_\nu^{\varkappa}(\theta) = s_\nu^{-\varkappa}(\theta)[s_\nu^{\varkappa}(\theta)]^{-1}$$

Now taking account of (2.4) and (2.7), we obtain the following expansion formula (/2/, p. 59) for the arbitrary function $f(u)$ from the sufficiently general class (the necessary constraints on the function $f(u)$ are mentioned in /2/)

$$f(u) = \int_0^\infty \chi_\nu^{\varkappa}(u, \theta)\rho(\tau)\tau\, d\tau \int_0^\infty \chi_\nu^{\varkappa}(y, \theta)\operatorname{sh} y f(y)\, dy \tag{2.8}$$

Let us examine two special cases of (2.8) when $c = 0$ and $c = \infty$. We obtain (the upper and lower signs are taken corresponding to these cases)

$$f(u) = \int_0^\infty Ps_\nu^{\pm\varkappa}(\operatorname{ch} u, \theta)\rho_\pm(\tau)\tau\, d\tau \int_0^\infty Ps_\nu^{\pm\varkappa}(\operatorname{ch} y, \theta)\operatorname{sh} y f(y)\, dy \tag{2.9}$$

$$\rho_\pm(\tau) = \pi \operatorname{sh}(\pi\tau) s_\nu^{\varkappa}(\theta) s_\nu^{-\varkappa}(\theta)\{\mid\Gamma(1/2 \pm \varkappa + i\tau)\mid\mid \times \cos[\pi(\varkappa + i\tau)]\mid\}^{-2}$$

As $\theta \to 0$ we have

$$s_\nu^{\pm\varkappa}(\theta) \to 1, \quad Ps_\nu^{\pm\varkappa}(\operatorname{ch} u, \theta) \to P_\nu^{\pm\varkappa}(\operatorname{ch} u)$$

and (2.9) goes over into the well-known Meller integral transform formula /14/, p. 398).

To write the expansion formula (2.8) as it applies to the functions $S_\nu^{\varkappa(3)}(\operatorname{ch} u, \theta)$ we note that by taking (2.5) into account the representation (1.20) can be written in the form

$$\begin{aligned}&S_\nu^{\varkappa(3)}(\operatorname{ch} u, \theta) = A_\varkappa Ps_\nu^{\varkappa}(\operatorname{ch} u, \theta) - B_\varkappa Ps_\nu^{-\varkappa}(\operatorname{ch} u, \theta), \\ &0 < u < \infty \\ &A_\varkappa = A_\varkappa(\tau, \theta) = E_\varkappa(\tau)[\sin(2\pi\delta)\overline{K_\nu^{\varkappa}(\theta)} + i\sin(2\pi\bar\delta)K_\nu^{\varkappa}(\theta)] \\ &B_\varkappa = B_\varkappa(\tau, \theta) = E_\varkappa(\tau)[\sin(2\pi\delta)\overline{K_\nu^{\varkappa}(\theta)}\Gamma(1 - 2\delta)(\Gamma(2\bar\delta))^{-1} + \\ &\quad i\sin(2\pi\bar\delta)K_\nu^{\varkappa}(\theta)\Gamma(1 - 2\bar\delta)(\Gamma(2\delta))^{-1}] \\ &E_\varkappa(\tau) = [2\sin(\pi\varkappa)\operatorname{sh}(\pi\tau)]^{-1}\exp(2\pi\tau)\end{aligned}$$

It therefore follows that for $c = B_\varkappa s_\nu^{\varkappa}(\theta)[A_\varkappa s_\nu^{-\varkappa}(\theta)]^{-1}$ the function $\chi_\nu^{\varkappa 3}(u, \theta)$ agrees with the function $S_\nu^{\varkappa(3)}$ (cosh u,θ) apart from a factor. Taking the last of (2.8) into account, we obtain

$$f(u) = \int_0^\infty S_\nu^{\varkappa(3)}(\operatorname{ch} u, \theta)\sigma(\tau)\tau\, d\tau \int_0^\infty S_\nu^{\varkappa(3)}(\operatorname{ch} y, \theta)\operatorname{sh} y f(y)\, dy \tag{2.10}$$

$$\begin{aligned}\sigma(\tau) = \pi \operatorname{sh}(\pi\tau) s_\nu^{\varkappa}(\theta) s_\nu^{-\varkappa}(\theta)\{[B_\varkappa^2 \mid \Gamma(1/2 - \varkappa + i\tau)\mid^2 + \\ A_\varkappa^2 \mid \Gamma(1/2 + \varkappa + i\tau)\mid^2 \mid \cos[\pi(\varkappa + i\tau)]\mid^2 - \\ 2\pi\cos(\pi\varkappa)\operatorname{ch}(\pi\tau) A_\varkappa B_\varkappa\}^{-1}, \quad \nu = -1/2 + i\tau\end{aligned}$$

3. We apply the results obtained to the following contact problem. Let two identical stamps, in the shape of two half-planes ω in planform, be displaced only translationally in the vertical direction under the action of definite moments and vertical forces $p_0(x, y)$ distributed along their upper faces, which have the finite resultant P, and be impressed in the half-space $z < 0$. We examine this problem in the non-linear steady creep theory formulation, when the material of the half-space is subjected to the power-law dependence $\sigma_i = K\varepsilon_i^h$ $(0 < h \leqslant 1)$ [3]. Here σ_i and ε_i are, respectively, the stress and strain rate intensities, while

K and h are physical constants.

Confining ourselves to the generalized principle of superposition of the displacements /3/, the problem mentioned can be formulated mathematically with respect to the unknown contact stresses $p(x, y)$ in the form of the integral equation

$$\iint_\omega \frac{p(\xi,\eta)\,d\xi\,d\eta}{[(x-\xi)^2+(y-\eta)^2]^{1-h/2}} = \left\{\frac{\delta - f(x,y)}{A}\right\}^h, \quad A = c(h)K^{-\gamma}, \quad \gamma = \frac{1}{h} \tag{3.1}$$
$$\iint_\omega p(\xi,\eta)\,d\xi\,d\eta = P \quad (P < \infty)$$

where δ is the settling of the stamps, $f(x, y)$ is a function characterizing their bases, $c(h)$ is a definite constant, where $c(3/2) = 0$, $c(1) = 1/(4\pi)$, and $c(h) > 0$ for $3/2 < h \leqslant 1$. We consider this last constraint on h to be conserved everywhere later.

By comparing the asymptotic forms of the left and right sides of (3.1) as $x^2 + y^2 \to \infty$, we find that there should be

$$f(x, y) \backsim \delta - AP^\gamma (x^2+y^2)^{1/2-\gamma}, \quad x^2+y^2 \to \infty$$

from which δ is actually determined.

Furthermore, we turn to dimensionless quantities in the integral equation (3.1) and then apply the Fourier integral transform in the variable x to both sides. We consequently arrive at the following one-dimensional integral equation

$$\left(\int_{-\infty}^{-1} + \int_1^\infty\right) \frac{K_\mu(|s||\bar{y}-\bar{\eta}|)}{|\bar{y}-\bar{\eta}|^\mu}\varphi_s(\bar{\eta})\,d\bar{\eta} = f_s(\bar{y}), \quad |\bar{y}| > 1 \tag{3.2}$$
$$\varphi_s(y) = \int_{-\infty}^{\infty} \varphi(\bar{x}, \bar{y}) e^{i\bar{x}s}\,d\bar{x}, \quad g_s(\bar{y}) = \int_{-\infty}^{\infty} g(\bar{x}, \bar{y}) e^{is\bar{x}}\,d\bar{x}$$
$$f_s(\bar{y}) = \pi^{-1/2} 2^{\mu-1}\Gamma(\mu + 1/2)|s|^{-\mu} g_s(\bar{y})$$
$$x, y;\ \xi, \eta = a\bar{x}, a\bar{y};\quad a\bar{\xi}, a\bar{\eta},\ \mu = (1-h)/2$$
$$\varphi(\bar{x}, \bar{y}) = A^h p(a\bar{x}, a\bar{y}), \quad g(\bar{x}, \bar{y}) = [\delta_0 - f_0(\bar{x}, \bar{y})]^h$$
$$\delta_0 = \delta/a, \quad f_0(\bar{x}, \bar{y}) = f(a\bar{x}, a\bar{y})/a$$

Having solved (3.2), the Fourier transform $w_s(\bar{y})$ of the generalized vertical displacements $w(\bar{x}, \bar{y}) = [-a^{-1}u_z'(a\bar{x}, a\bar{y})]^h$ ($u_z(x, y)$ are the true vertical displacements) of the half-space boundary points outside the stamps will be expressed by the formula

$$w_s(\bar{y}) = \left(\int_{-\infty}^{-1} + \int_1^\infty\right) \frac{K_\mu(|s||\bar{y}-\bar{\eta}|)}{|\bar{y}-\bar{\eta}|^\mu}\psi_s(\bar{\eta})\,d\bar{\eta}, \quad |\bar{y}| < 1 \tag{3.3}$$
$$\psi_s(\bar{y}) = \sqrt{\pi}\, 2^{1-\mu}|s|^\mu \varphi_s(\bar{y})/\Gamma(\mu + 1/2)$$

Now, considering the symmetric case of stamp loading, we represent the solution of (3.2) in the form

$$\varphi_s(\bar{y}) = (\bar{y}^2 - 1)^{-\varkappa/2}\int_0^\infty \Phi_s^\varkappa(\tau) S_\nu^{\varkappa(3)}(\bar{y}, \theta)\,d\tau, \quad \bar{y} > 1 \tag{3.4}$$

where $\Phi_s^\varkappa(\tau)$ is an as yet unknown function. We substitute this expression for $\varphi_s(\bar{y})$ into (3.2), interchange the order of integration, and use the relationship (1.19) with the plus sign. We will have

$$(\bar{y}^2 - 1)^{\varkappa/2}\int_0^\infty \lambda_{\nu,\varkappa}^+ \Phi_s^\varkappa(\tau) S_\nu^{\varkappa(3)}(\bar{y}, \theta)\,d\tau = f_s(\bar{y}), \ \bar{y} > 1$$

We hence obtain by means of (2.10), in which we replace $\operatorname{ch} y$ by $\bar{y}$,

$$\Phi_s^\varkappa(\tau) = (\lambda_{\nu,\varkappa}^+)^{-1}\sigma(\tau)\tau \int_0^\infty (\bar{y}^2 - 1)^{-\varkappa/2} f_s(\bar{y}) S_\nu^{\varkappa(3)}(\bar{y}, \theta)\,d\bar{y} \tag{3.5}$$

In the symmetric case, the solution of (3.2) is therefore given by (3.4)-(3.5).

Finally, substituting $\varphi_s(\bar{y})$ from (3.4) into (3.3) and taking (1.21) into account with the plus sign, we find $(0 < \bar{y} < 1)$

$$w_s(\bar{y}) = (1 - \bar{y}^2)^{\varkappa/2}\int_0^\infty C_\nu^\varkappa(s, \theta)\Phi_s^\varkappa(\tau) H_{\nu,\varkappa}^+(\arccos \bar{y})\,d\tau$$
$$C_\nu^\varkappa(s, \theta) = \sqrt{\pi}\, 2^{\varkappa+1/2} h_{\nu,\varkappa}^+ |s|^{1/2-\varkappa}[\Gamma(1-\varkappa)]^{-1}$$

It should be emphasized that in the selected case it is necessary to set $a = 1$ formally into the expressions for $\lambda_{\nu,\varkappa}^{+}$ and $h_{\nu,\varkappa}^{+}$ from (1.19) and (1.21).

We note that the results obtained here can be applied to a fairly large number of contact and mixed problems of elasticity theory as well as to modified mixed problems of mathematical physics. The need to tabulate the functions $S_\nu^{\varkappa(3)}(x,\theta)$ $(0 \leqslant x < \infty)$ and $G_{\nu,\varkappa}^{\pm}(\arccos x)$ $(|x|<1)$ arises here; this can be achieved by using continued fractions /10-12/.

REFERENCES

1. RAKOV A. KH. and RVACHEV V.L., A contact problem of elasticity theory for a half-space whose elastic modulus is a power-law function of the depth, Dokl. Akad. Nauk UkrSSR, Ser. A, No.3, 1961.
2. TITCHMARSH E.C., Eigenfunction Expansions Associated with Second-Order Differential Equations /Russian translation/, Vol. 1, Izdat. Inostr. Lit, Moscow, 1960.
3. ARUTIUNIAN N. KH., A plane contact problem of creep theory, PMM, Vol.23, No.5, 1959.
4. POPOV G. YA., Contact Problems for a Linearly Deformable Base, Vishcha Shkola, Kiev-Odessa, 1982.
5. POPOV G. YA., Elastic Stress Concentration around Stamps, Slits, Thin Inclusions, and Reinforcements, Nauka, Moscow, 1982.
6. BELWARD J.A., On the relationship of some Fredholm integral equations of the first kind to a family of boundary value problems, J. Math. Analysis and Appl. Vol.48, No.1, 1974.
7. ALEKSANDROV V.M., KOVALENKO E.V. and MKHITARYAN S.M., On a method of obtaining spectral relationships for integral operators of mixed problems of the mechanics of continuous media, PMM, Vol.46, No.6, 1982.
8. MKHITARYAN S.M., On two spectral relationships for integral operators in a semi-infinite interval and their application to mixed problems, Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, No.1, 1983.
9. MKHITARYAN S.M., On certain spectral relationships associated with the Carlemann integral equation and their applications to contact problems, PMM, Vol.47, No.2, 1983.
10. BATEMAN H. and ERDELYI A., Higher Transcendental Functions /Russian translation/, Nauka, Moscow, 1967.
11. MEIXNER J., Klassifikation, Bezeichnung und Eigenschaften der Sphäroidfunktionen, Math. Nachr., Vol.5, 1951.
12. MEIXNER J. and SCHAFKE F.W., Mathieusche Funktionen und Sphäroidfunktionen mit Anwendungen auf physikalische und technische Probleme, Springer, Berlin-Heidelberg, 1954.
13. BATEMAN H. and ERDELYI A., Higher Transcendental Functions /Russian translation/, Vol.1, Nauka, Moscow, 1973.
14. MAGNUS W., OBERHETTINGER F. and SONI R.P., Formulas and Theorems for the Special Functions of Mathematical Physics, Springer, Berlin-Heidelberg-New York, 1966.

Translated by M.D.F.

PMM U.S.S.R.,Vol.48,No.5,pp.618-626,1984
Printed in Great Britain
0021-8928/84 $10.00+0.00

ASYMPTOTIC SOLUTION OF THREE-DIMENSIONAL PROBLEMS OF THE THEORY OF ELASTICITY OF EXTENDED PLANE SEPARATION CRACKS*

R.V. GOL'DSHTEIN, A.V. KAPTSOV, and L.B. KOREL'SHTEIN

A solution of three-dimensional elasticity theory problems for separation cracks occupying a plane domain with one characteristic dimension much smaller than the other is constructed by the method of matched asymptotic expansions (cracks that are extended along a certain plane curve). The appropriate terms of the expansion of the solution in a small parameter characterizing the extent of the crack are constructed using an integro-differential equation in the displacement of points of the crack surface. For cracks that are extended along a line, the representation of the integro-differential equation in terms of a two-dimensional Fourier transform is used, which substantially simplifies the calculation. In the general case, the expansion is executed directly in the equation written in x-space. The asymptotic expansion constructed is valid in the middle part of the

**Prikl.Matem.Mekhan.*,Vol.48,5,854-863,1984